

On Quantum Deformations of $d = 4$ Conformal Algebra ^{1 2}

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Abstract

Three classes of classical r matrices for $sl(4, C)$ algebra are constructed in quasi-Frobenius algebra approach. They satisfy CYBE and are spanned respectively on 8,10,12 generators. The $o(4, 2)$ reality condition can be imposed only on the eight dimensional r matrices with dimension-full deformation parameters. Contrary to the Poincaré algebra case, it appears that all deformations with a mass-like deformation parameter (κ - deformations) are described by classical r -matrices satisfying CYBE.

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1 Introduction.

In four dimensions the deformations of $o(4, 2)$ algebra describing $d = 4$ conformal algebra can be obtained by considering the deformations of complexified $d = 4$ conformal algebra $sl(4, C)$ and then by taking into account the restrictions imposed by the reality conditions (which define the deformed $o(4, 2)$ algebra as a real form of deformed complex $sl(4, C)$). It appears that the reality conditions are quite restrictive. In [1] there were classified all the real forms of Drinfeld-Jimbo deformation $U_q(sl(4, C))$ of complexified $d = 4$ conformal algebra. It appears that for standard $*$ -Hopf algebra, with $*$ -operation being anti-automorphism of algebra and an automorphism of coalgebra there exist only two real forms of the Drinfeld-Jimbo deformations of $U_q(sl(4, C))$, one for q real and second for $|q| = 1$, providing the q -deformations $U_q(o(4, 2))$ of $d = 4$ conformal algebra.

In this paper we describe deformations of $sl(4, C)$ which admit the structure of the real deformed $d = 4$ conformal algebra. To give the classification of quantum deformations, we shall discuss here the classical $sl(4)$ r -matrices. We present results which are an essential extension of those obtained in [4]. Firstly in sect. 2 we consider the mathematical (Cartan-Weyl for $sl(4)$) as well as physical (complex-conformal $o(4, 2; C)$) basis and then by providing two possible real forms from [1] we introduce respective physical real bases of $d = 4$ conformal algebra. In sect. 3 using the result of [7] and the techniques presented by Alexeevsky, Perelomov [7] and Stolin [9, 10] we describe three classes of classical r -matrices for $sl(4, C)$ spanned on 8, 10, 12 dimensional subalgebras. Considering the $o(4, 2)$ real forms we show that only one class (eight dimensional) permits the $o(4, 2)$ reality conditions. It appears that these new classical r -matrices are span by the generators of $d = 4$ Weyl algebra (Poincaré generators and dilatations) possibly transformed by Weyl reflections. In Sect. 4 we shall present some remarks and conclusions (in particular, concerning the structure relations of the κ -deformed $d = 4$ conformal algebra by applying the twist transformation proposed by Kulish, Lyakhovsky and Mudrov [11]).

2 Cartan-Weyl basis and its real forms.

One can write the complex $sl(4; C)$ algebra in Cartan Weyl basis e_{AB} ($A, B = 1, \dots, 4$); where the choice of indices (A, B) is taken from the position of non-vanishing entry in the 4×4 fundamental matrix representation. In particular the diagonal elements $h_1 = e_{11} - e_{22}$, $h_2 = e_{22} - e_{33}$, $h_3 = e_{33} - e_{44}$ describe three commuting Cartan generators, and the simple root generators $e_1 = e_{12}$, $e_2 = e_{23}$ and $e_3 = e_{34}$ describe Cartan-Chevalley basis. The composite roots extending Cartan-Chevalley basis to Cartan-Weyl basis are described by the formulae: $e_4 \equiv e_{13} = [e_{12}, e_{23}]$, $e_5 \equiv e_{24} = [e_{23}, e_{34}]$, $e_6 \equiv e_{14} = [e_{12}, e_{24}] = [e_{13}, e_{34}]$. Classical $sl(4)$ algebra is generated by the relations satisfied by the Cartan-Weyl basis generators

$e_{\pm A}, h_A$ ($A = 1, 2, \dots, 6$; $h_4 = h_1 + h_2$, $h_5 = h_2 + h_3$, $h_6 = h_1 + h_2 + h_3$):

$$\begin{aligned} [h_A, e_{\pm B}] &= \pm \alpha_{AB} e_{\pm B}, \\ [e_i, e_{-j}] &= \delta_{ij} h_j, \quad i = 1, 2, 3, \\ [e_a, e_{-a}] &= h_a, \quad a = 4, 5, 6, \end{aligned} \quad (1)$$

Remaining relations of Cartan-Weyl basis of $U(sl(4, C))$ are generated by Serre relations and the above definitions. In order to describe the real forms of complex Lie algebra $sl(4)$ we consider involutive anti-automorphisms $x \rightarrow x^*$ of $U(sl(4, C))$ such that for any $x, y \in U(sl(4, C))$

$$(xy)^* = y^* x^*, \quad (\mu X + \lambda y)^* = \mu^* x^* + \lambda^* y^*, \quad \mu, \lambda \in C \quad (2)$$

There are the following inequivalent real forms describing by means of the reality condition $x = x^*$ the real $o(4, 2)$ algebra [1] ($j = 1, 2, 3$)

$$h_j^* = -h_{4-j}, \quad e_{\pm j}^* = e_{\pm(4-j)} \quad (3)$$

$$h_j^* = h_j, \quad e_{\pm j}^* = \epsilon_j e_{\mp j} \quad (4)$$

with three nonequivalent choices of $(\epsilon_1, \epsilon_2, \epsilon_3)$: $(1, -1, 1)$, $(-1, 1, -1)$ and $(-1, -1, -1)$.

Let us observe that $sl(4, C) \simeq o(4, 2, C)$ and the generators $M_{RQ} = -M_{QR}$ ($P, Q = 0, 1, 2, 3, 4, 5$) of $o(4, 2; C)$ ($\eta_{AB} = \text{diag}(-1, 1, 1, 1, 1, -1)$) satisfy the relations

$$[M_{PQ}, M_{RS}] = \eta_{PS} M_{QR} - \eta_{PR} M_{QS} + \eta_{QR} M_{PS} - \eta_{QS} M_{PR} \quad (5)$$

We extend the Lorentz generators $M_{\mu\nu} = (M_i = \frac{1}{2} \epsilon_{ijk} M_{jk}, L_i = M_{i0})$; $\mu, \nu = 0, 1, 2, 3$, to $d = 4$ conformal algebra generators as follows:

$$P_\mu = (M_{4\mu} + M_{5\mu}) \quad K_\mu = (M_{5\mu} - M_{4\mu}) \quad D = M_{45} \quad (6)$$

The reality conditions lead to the following two ways of defining real $d = 4$ conformal algebra generators in terms of Cartan-Weyl basis of $\hat{g} = sl(4, C)$; $\hat{g} = B^- \oplus H \oplus B^+$, where (B^+, H) , (B^-, H) are two Borel subalgebra and $H = (h_1, h_2, h_3)$ describe Cartan subalgebra.

i) The reality condition $(B^\pm)^* \subset B^\pm$.

$$\begin{aligned} M_+ &= e_1 + e_{-3} & M_- &= -(e_3 + e_{-1}) & M_3 &= \frac{i}{2}(h_1 - h_3) \\ L_+ &= i(e_{-3} - e_1) & L_- &= -i(e_3 - e_{-1}) & L_3 &= \frac{i}{2}(h_1 + h_3) \\ P_1 &= -(e_4 + e_5) & P_2 &= i(e_4 - e_5) & P_3 &= i(e_2 - e_6) \\ K_1 &= e_{-4} - e_{-5} & K_2 &= i(e_{-4} + e_{-5}) & K_3 &= i(e_{-2} - e_{-6}) \\ P_0 &= -i(e_2 + e_6) & K_0 &= i(e_{-2} + e_{-6}) & D &= \frac{1}{2}(h_1 + 2h_2 + h_3) \end{aligned} \quad (7)$$

where $M_\pm = M_1 \pm iM_2$, $L_\pm = L_1 \pm iL_2$. We see that the Cartan subalgebra H is described by the non compact algebra (M_3, L_3, D) , and under the $*$ -operation

the operators are real.

ii) The reality condition $(B^\pm)^* \subset B^\mp$. This reality condition can not be applied to the solutions of the CYBE for $sl(4)$ with a dimension of solution no less than eight. Hence the assignment of the conformal generators for this case will be not needed in further considerations and it is omitted here.

It appears that the Cartan subalgebra H is described by the compact Abelian subgroup $(M_{12} = M, M_{34} = \frac{1}{2}(P_3 - K_3), M_{50} = \frac{1}{2}(P_0 + K_0))$. The choices of the generators can be modified if we take into consideration the discrete group of Weyl reflections, which preserve the Lie-algebra relations. There are three basic Weyl reflections $\sigma_1, \sigma_2, \sigma_3$ describing the automorphism of $sl(4, C)$ Lie algebra. For example explicit relations defining σ_1 are the following:

$$\begin{aligned} \sigma_1(e_{\pm 1}) &= (a_1)^{\mp 1} e_{\mp 1} \quad , \quad \sigma_1(e_{\pm 2}) = (a_4)^{\pm 1} e_{\pm 4} \quad , \quad \sigma_1(e_{\pm 3}) = (a_3)^{\pm 1} e_{\pm 3} \\ \sigma_1(e_{\pm 4}) &= (a_2)^{\pm 1} e_{\pm 2} \quad , \quad \sigma_1(e_{\pm 5}) = (a_6)^{\pm 1} e_{\pm 6} \quad , \quad \sigma_1(e_{\pm 6}) = (a_5)^{\pm 1} e_{\pm 5} \end{aligned} \quad (8)$$

$a_4 = a_1 a_2$ $a_5 = a_2 a_3$ $a_6 = a_1 a_2 a_3$. There exists also the isomorphism of Dynkin diagram $(\alpha_1 \leftrightarrow \alpha_3)$ which implies the following isomorphism of $sl(4; C)$ Lie algebra:

$$\beta(e_{\pm 1}) = e_{\pm 3} \quad , \quad \beta(e_{\pm 2}) = e_{\pm 2} \quad , \quad \beta(e_{\pm 4}) = e_{\pm 5} \quad , \quad \beta(e_{\pm 6}) = e_{\pm 6} \quad (9)$$

Any product of Weyl reflections is again an isomorphism of $sl(4; C)$, but not all these isomorphisms commute with the $*$ -operations defining real forms. The condition

$$\sigma_{i_1 \dots i_k} \cdot * = * \cdot \sigma_{i_1 \dots i_k} \quad (10)$$

is necessary for defining the restriction of Weyl reflections to $o(4, 2)$ algebra. We obtain

- i) for the $*$ -operation $(B^\pm)^* \subset B^\pm$ the involutions σ_2 , $\sigma_1 \sigma_3 = \sigma_3 \sigma_1$, β are also isomorphisms of real algebra $o(4, 2)$ provided that $b_1^* = b_3$, $b_2^* = b_2$.
- ii) for the $*$ -operation $(B^\pm)^* \subset B^\mp$ we obtain the following isomorphisms of $o(4, 2)$: σ_2, β . Provided that $b_i^* b_i = 1$.

3 Classical r -matrices for $sl(4)$ and $o(4, 2)$ reality conditions

We shall consider the antisymmetric solutions of the CYBE i.e.

$$<< r, r >> = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \quad , \quad r \in \hat{g} \wedge \hat{g}, \quad (11)$$

where $<< r, r >>$ denotes Schouten bracket ($<< r, r >> \in \hat{g} \otimes \hat{g} \otimes \hat{g}$). In order to construct new solutions we shall apply the quasi-Frobenius algebra approach, using the following definitions and results [13, 12, 9, 10]:

◇ Lie algebra \hat{g} is quasi-Frobenius if there exists a skew-symmetric bilinear form $B : \hat{g} \wedge \hat{g} \rightarrow C$ such that for all $x, y, z \in \hat{g}$

$$B([x, y], z) + B([y, z], x) + B([z, x], y) = 0. \quad (12)$$

Let $b_{ij} = \langle e_i^* \otimes e_j^*, B \rangle$ then $r = r^{ij} e_i \wedge e_j$, where $r^{ij} b_{jk} = \delta_k^i$ satisfies CYBE

◇◇ If the bilinear form B is determined by a functional g_B^* on \hat{g} such that

$$B(x, y) = \langle g_B^*, [x, y] \rangle, \quad (13)$$

then \hat{g} is called Frobenius algebra.

◇◇◇ The classification of classical r -matrices (obtained in this way) can be reduced to the classification of quasi - Frobenius algebras which in turn, are even dimensional and can be identified with a set of parabolic subalgebras.

For $\hat{g} = sl(4, C)$ we can distinguish three relevant families of the parabolic subalgebras spanning the respective classical r -matrices. Let $B_+ = (h_i, e_A)$; $i = 1, 2, 3$; $A = 1, \dots, 6$ denotes a Borel subalgebra, then we have explicitly the following classification of classical r -matrices:

d=12. Parabolic subalgebra $P_{(-2, -3)} = (B_+, e_{-2}, e_{-3})$. In this case one obtains the one-parameter generalization of the solution given by Gerstenhaber and Giquinto [8]

$$r_{(12)} = \frac{1}{4}(3h_1 + 2h_2 + h_3) \wedge e_1 + \frac{1}{4}(h_1 + 2h_2 + 3h_3) \wedge e_3 + e_4 \wedge e_{-2} + e_6 \wedge e_{-5} + \lambda \left(\frac{1}{2}(h_1 + 2h_2 + h_3) \wedge e_2 + (e_4 + e_5) \wedge e_{-3} \right) \quad (14)$$

This solution of the CYBE has the following properties: parameter λ is arbitrary (it has inverse of mass dimension), each part of r satisfies CYBE separately, it does not permit the restriction of $sl(4, C)$ to real $o(4, 2)$

d=10. Parabolic subalgebras $P_{(j)} = (B_+, e_{-j})$, $j = 1, 2, 3$. We have to consider here three separate sets of nonsingular functionals:

i) Parabolic subalgebra P_1 .

$$\begin{aligned} g_{1a}^* &= e_5^* + e_4^* + e_1^* & g_{1d}^* &= e_6^* + e_2^* + e_{-1}^* \\ g_{1b}^* &= e_5^* + e_4^* + e_3^* & g_{1e}^* &= e_6^* + e_2^* + e_1^* + e_3^* \\ g_{1c}^* &= e_6^* + e_2^* + e_3^* \end{aligned} \quad (15)$$

They yield the following r matrices:

$$\begin{aligned}
r_{1a}^{(10)} &= -e_2 \wedge e_3 + e_6 \wedge e_{-1} + \frac{1}{2}(e_1 + e_3) \wedge (h_1 + h_3) \\
&\quad + \frac{1}{4}e_4 \wedge (h_1 + 2h_2 - h_3) + \frac{1}{4}e_5 \wedge (h_1 + 2h_2 + 3h_3) \\
r_{1b}^{(10)} &= -e_1 \wedge e_2 + e_6 \wedge e_{-1} + \frac{1}{2}(e_1 + e_3) \wedge (h_1 + h_3) \\
&\quad + \frac{1}{4}e_4 \wedge (3h_1 + 2h_2 + h_3) + \frac{1}{4}e_5 \wedge (-h_1 + 2h_2 + 3h_3) \\
r_{1c}^{(10)} &= -e_1 \wedge e_5 + e_4 \wedge e_{-1} + \frac{1}{2}(e_3 + e_{-1}) \wedge (-h_1 + h_3) \\
&\quad + \frac{1}{4}e_2 \wedge (-h_1 + 2h_2 + h_3) + \frac{1}{4}e_6 \wedge (3h_1 + 2h_2 + h_3) \\
r_{1d}^{(10)} &= -e_1 \wedge e_5 + e_3 \wedge e_4 + \frac{1}{2}(e_3 + e_{-1}) \wedge (-h_1 + h_3) \\
&\quad + \frac{1}{4}e_2 \wedge (h_1 + 2h_2 - h_3) + \frac{1}{4}e_6 \wedge (h_1 + 2h_2 + 3h_3) \\
r_{1e}^{(10)} &= -\frac{1}{2}e_1 \wedge e_2 + e_6 \wedge e_{-1} + \frac{1}{4}(e_1 + e_3) \\
&\quad \wedge (h_1 + h_3) - \frac{1}{2}e_2 \wedge (e_3 + e_4 - e_5) \\
&\quad + \frac{1}{2}e_4 \wedge (h_1 + h_2) + \frac{1}{2}e_5 \wedge (h_2 + h_3)
\end{aligned} \tag{16}$$

ii) Parabolic subalgebra $P_{(2)}$. We have shown explicitly by considering the most general ansatz that there does not exist a Frobenius algebra structure on $P_{(2)}^*$

iii) Parabolic subalgebra $P_{(3)}$. Nonsingular functionals:

$$\begin{aligned}
g_{3a} &= e_5^* + e_4^* + e_1^* & g_{3d} &= e_6^* + e_2^* + e_{-3}^* \\
g_{3b} &= e_5^* + e_4^* + e_3^* & g_{3e} &= e_5^* + e_4^* + e_1^* + e_3^* \\
g_{3c} &= e_6^* + e_2^* + e_{+1}^*
\end{aligned} \tag{17}$$

yielding the following r matrices:

$$\begin{aligned}
r_{3a}^{(10)} &= -e_2 \wedge e_3 - e_6 \wedge e_{-3} + \frac{1}{2}(e_1 + e_3) \wedge (h_1 + h_3) \\
&\quad + \frac{1}{4}e_4 \wedge (h_1 + 2h_2 - h_3) + \frac{1}{4}e_5 \wedge (h_1 + 2h_2 + 3h_3) \\
r_{3b}^{(10)} &= -e_1 \wedge e_2 - e_6 \wedge e_{-3} + \frac{1}{2}(e_1 + e_3) \wedge (h_1 + h_3) \\
&\quad + \frac{1}{4}e_4 \wedge (3h_1 + 2h_2 + h_3) + \frac{1}{4}e_5 \wedge (-h_1 + 2h_2 + h_3) \\
r_{3c}^{(10)} &= e_3 \wedge e_4 - e_5 \wedge e_{-3} - \frac{1}{2}(e_1 + e_3) \wedge (-h_1 + h_3) \\
&\quad + \frac{1}{4}e_2 \wedge (h_1 + 2h_2 - h_3) + \frac{1}{4}e_6 \wedge (h_1 + 2h_2 + 3h_3) \\
r_{3d}^{(10)} &= -e_1 \wedge e_5 + e_3 \wedge e_4 - \frac{1}{2}(e_1 + e_{-3}) \wedge (-h_1 + h_3) \\
&\quad + \frac{1}{4}e_2 \wedge (-h_1 + 2h_2 + h_3) + \frac{1}{4}e_6 \wedge (3h_1 + 2h_2 + h_3) \\
r_{3e}^{(10)} &= -\frac{1}{2}e_1 \wedge e_5 - e_6 \wedge e_{-3} + \frac{1}{4}(e_1 + e_3) \wedge (-h_1 + h_3) \\
&\quad - \frac{1}{2}e_2 \wedge (e_3 + e_4 - e_5) + \frac{1}{2}e_4 \wedge (h_1 + h_2) + \frac{1}{2}e_5 \wedge (h_2 + h_3)
\end{aligned} \tag{18}$$

It can be shown that all such generated 10-dimensional classical r -matrices do not permit the $o(4, 2)$ reality conditions. It is that because σ_2 commutes with the $*$ and $(\sigma_2 \otimes \sigma_2)r_3^{(10)} = r_1^{(10)}$, but these r -matrices are not compatible i.e. $\langle\langle r_1^{(10)}, r_3^{(10)} \rangle\rangle \neq 0$.

d=8. Here we have the Borel subalgebra B_+ .

$$r_1^{(8)} = e_4 \wedge e_3 - e_5 \wedge e_1 + ah_2 \wedge e_6 + h_6 \wedge e_6 \tag{19}$$

Taking into account that above classical r -matrix is real under the $*$ -operation (3) and using the Weyl automorphism commuting with this $*$ -involution we obtain

another form of the d=8 solution:

$$r_2^{(8)} = (\sigma_2 \otimes \sigma_2) \circ r_1^{(8)} = [(\sigma_1 \sigma_3) \otimes (\sigma_1 \sigma_3)] \circ r_1^{(8)} = e_5 \wedge e_{-3} - e_4 \wedge e_{-1} + h_2 \wedge e_2 + ah_6 \wedge e_2 \quad (20)$$

Let us note that in the physical basis above r -matrices are spanned on generators of the $d = 4$ Weyl subalgebra (M_i, L_i, P_μ, D) .

4 Final remarks.

In this paper we have considered r -matrices satisfying CYBE. From their scaling properties in the physical basis and the fact that invariant three form for $o(4, 2)$; $I = e_{ij} \wedge e_{jk} \wedge e_{ki} \sim M_A^B \wedge M_B^C \wedge M_C^A$ is scale invariant it follows that every r -matrix giving dimension-full deformation (κ -deformation) satisfies CYBE.

We do not present here the description of the complete κ -deformed algebra. Using the results of the work [11] one can obtain the coproduct applying to the Δ_0 the twist F of the form: $\Delta_F(x) = F \circ \Delta_0(x) \circ F^{-1}$, where $F = \exp(h_6 \otimes \sigma(e_6)) \cdot \exp(2\lambda e_1 \otimes e_5 \cdot e^{-2\sigma(e_6)}) \cdot \exp(2\lambda e_4 \otimes e_3 e^{-2\sigma(e_6)})$ and $\sigma(e_6) = -\frac{1}{2} \ln(1 - 2\lambda e_6) \sim \ln(1 - \frac{i}{\kappa}(P_0 + P_3))$. It will be given in a forthcoming paper of the present authors.

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